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SOLUTION OF THE PERTURBATION PROBLEM FOR A SHEAR FLOW WITH NONMONOTONIC VELOCITY PROFILE

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The problem of disturbances in a given plane flow of an ideal incompressible fluid, which corresponds to a shear flow with a horizontal free boundary, is considered. In the long-wave approximation, the disturbed flow is described by a solution of the Cauchy problem for a linear system of integrodifferential equations, which in the absence of shear of the velocity vector coincides with the well-known linearized equations of the shallow-water theory. An explicit solution of the Cauchy problem is obtained for this system of equations.

1. Statement of the Problem. Let us consider the initial boundary-value problem with a free boundary:

$$
\begin{gathered}
\rho\,(U_T+UU_X+VU_Y)+p_X=0 \qquad (0\leqslant Y\leqslant H(X,T)),\\
U_X+V_Y=0,\quad p_Y=-\rho g,\quad H_T+(\int\limits_0^H U\,dY)_X=0, \qquad (1.1)\\
p(X,H(X,T),T)=0,\quad V(X,0,T)=0,\quad U(X,Y,0)=U_0(X,Y),\quad H(X,0)=H_0(X).
\end{gathered}
$$

Problem (1.1) describes, in the long-wave approximation, a plane-parallel vortex flow of a layer of a homogeneous heavy fluid of depth $H=H(X,T)$ above an even bottom $Y=0$. Here U and V are the components of the velocity vector of the fluid; p is the pressure; ρ is the density ($\rho=\text{const}$); g is the free fall acceleration; and $U_0(X,Y)$ and $H_0(X)$ are prescribed functions.

It was shown [1] that problem (1.1) is reduced to the Cauchy problem for the system of integrodifferential equations

$$
\begin{gathered}
u_t+uu_x+g\int\limits_0^1 h_x d\nu=0,\quad h_t+(uh)_x=0,\quad 0\leqslant\lambda\leqslant 1,\\
u(x,0,\lambda)=U_0(x,\lambda H_0(x)),\quad h(x,0,\lambda)=H_0(x),
\end{gathered}
\qquad (1.2)
$$

where $u(x,t,\lambda)=U(x,\Phi(x,t,\lambda),t)$; $h(x,t,\lambda)=\Phi_\lambda(x,t,\lambda)$; the function $\Phi(x,t,\lambda)$ is defined by solving the problem

$$
\Phi_t+(\int\limits_0^\Phi U(x,Y,t)\,dY)_x=0,\qquad \Phi(x,0,\lambda)=\lambda H_0(x).
$$

The coordinate surfaces $\lambda=\text{const}$ are contact surfaces; $\lambda=0$ corresponds to the bottom, while $\lambda=1$, to a free surface. When $U_Y\equiv 0$ (which corresponds to a vortex-free flow in the long-wave approximation) Eqs. (1.2) coincide with the well-known equations of the shallow-water theory. The case of $U_Y\not\equiv 0$ corresponds to a vortex (shear) flow.

Teshukov [1, 2] gives a definition for the hyperbolicity of a system with operator coefficients and specified that the hyperbolicity conditions for Eqs. (1.2) are for the case of a monotonic velocity profile

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($U_Y \neq 0$). The hyperbolicity conditions of Eqs. (1.2) were obtained in [3] for a nonmonotonic velocity profile under the assumption that U_Y vanishes at a single point $Y_*(X,T)$, $0 < Y_*(X,T) < H(X,T)$ [in this case $U_{YY}(Y_*) \neq 0$].

In the present paper, an explicit solution of the Cauchy problem is obtained for the system of equations (1.2) linearized on the stationary solution $\mathbf{u} = (u_0(\lambda), h_0(\lambda))^t$ (a shear flow with a horizontal free boundary) (superscript t denotes transposition).

The linear Cauchy problem

$$u_t + u_0 u_x + g\int_0^1 h_x d\nu = 0, \quad h_t + h_0 u_x + u_0 h_x = 0, \quad u(x,0,\lambda) = u_2(x,\lambda), \quad h(x,0,\lambda) = h_2(x,\lambda) \qquad (1.3)$$

describes small perturbations of a shear flow.

We consider nonmonotonic velocity profiles $u_0(\lambda)$ satisfying the conditions

$$u_{0\lambda} > 0 \quad \text{at} \quad 0 < \lambda < \lambda_1, \quad u_{0\lambda} < 0 \quad \text{at} \quad \lambda_1 < \lambda < 1, \quad u_{0\lambda\lambda}(\lambda_1) \neq 0, \quad u_0(0) < u_0(1). \qquad (1.4)$$

As in [3], we introduce some auxiliary quantities [which are necessary for reducing the system of equations (1.3) to a characteristic form].

Let λ_2 be a point of the segment [0,1], where the equality $u_0(\lambda_2) = u_0(1)$ is satisfied. For any point λ of the segment $[\lambda_2, \lambda_1]$, we introduce the function $\lambda_s = \lambda_s(\lambda)$ ($\lambda_s \geqslant \lambda_1$) defined by the equality $u_0(\lambda) = u_0(\lambda_s(\lambda))$.

The smooth function ψ corresponds to a third-degree polynomial in the variable ν $Q(\nu,\lambda,\lambda_s,\psi)$ meeting the conditions

$$Q(\lambda,\lambda,\lambda_s,\psi) = \psi(\lambda), \quad Q(\lambda_s,\lambda,\lambda_s,\psi) = \psi(\lambda_s), \quad Q_\nu(\lambda,\lambda,\lambda_s,\psi) = \psi_\nu(\lambda), \quad Q_\nu(\lambda_s,\lambda,\lambda_s,\psi) = \psi_\nu(\lambda_s).$$

This polynomial is representable in the form

$$\begin{gathered} Q(\nu,\lambda,\lambda_s,\psi) = \frac{1}{2}(\psi(\lambda)+\psi(\lambda_s)) - \frac{1}{8}(\psi_\nu(\lambda)-\psi_\nu(\lambda_s))(\lambda-\lambda_s) + +(-\frac{1}{4}(\psi_\nu(\lambda)+\psi_\nu(\lambda_s)) \\ +\frac{3}{2}(\psi(\lambda)-\psi(\lambda_s))(\lambda-\lambda_s)^{-1})(\nu-\frac{1}{2}(\lambda+\lambda_s)) + \frac{1}{2}(\psi_\nu(\lambda)-\psi_\nu(\lambda_s))(\lambda-\lambda_s)^{-1}(\nu-\frac{1}{2}(\lambda+\lambda_s))^2 \\ +((\psi_\nu(\lambda))+\psi_\nu(\lambda_s))(\lambda-\lambda_s)^{-2} - 2(\psi(\lambda)-\psi(\lambda_s))(\lambda-\lambda_s)^{-3})(\nu-\frac{1}{2}(\lambda+\lambda_s))^3. \end{gathered}$$

The polynomial $Q(\nu,\lambda,\lambda_s)$ is constructed in such a way that the difference between the values $Q(\nu,\lambda,\lambda_s,\psi)$ and $\psi(\nu)$ in the vicinity of the point λ_1 (where $\lambda \to \lambda_s$) satisfy the relation $(\psi(\nu) - Q(\nu,\lambda,\lambda_s,\psi)) = O((\nu-\lambda_1)^4)$.

Let us introduce the polynomial $Q_{01}(\nu,\lambda,\lambda_s,\psi)$ by the formula

$$\begin{aligned} Q_1^0(\nu,\lambda,\lambda_s,\psi) = {} & \frac{1}{8}(\omega_0(\lambda)\psi(\lambda)-\omega_0(\lambda_s)\psi(\lambda_s))(\lambda-\lambda_s) \\ & +\frac{1}{4}(\omega_0(\lambda)\psi(\lambda)+\omega_0(\lambda_s)\psi(\lambda_s))(\nu-\frac{1}{2}(\lambda+\lambda_s)) \\ & -\frac{1}{2}((\omega_0(\lambda)\psi(\lambda)-\omega_0(\lambda_s)\psi(\lambda_s))(\lambda-\lambda_s)^{-1})(\nu-\frac{1}{2}(\lambda+\lambda_s))^2 \\ & -((\omega_0(\lambda)\psi(\lambda)+\omega_0(\lambda_s)\psi(\lambda_s))(\lambda-\lambda_s)^{-2})(\nu-\frac{1}{2}(\lambda+\lambda_s))^3, \end{aligned}$$

where $\omega_0(\lambda) = u_{0\nu}(\lambda)/h_0(\lambda)$.

We introduce the functionals $\delta(\lambda)$, $\delta'(\lambda)$, $P_{10}^0(\lambda)$ [$\lambda \in (0,\lambda_2)$], $P_{11}^0(\lambda)$ [$\lambda \in (0,\lambda_2)$], $P_0^0(\lambda)$ [$\lambda \in (\lambda_2,1)$], $P_1^0(\lambda)$ [$\lambda \in (\lambda_2,1)$] acting on the smooth trial function ψ following the rules:

$$(\delta(\lambda),\psi) = \psi(\lambda), \;\; (\delta'(\lambda),\psi) = -\psi_\nu(\lambda), \qquad (P_{10}^0(\lambda),\psi) = \int_0^1 \frac{h_0(\nu)(\psi(\nu)-\psi(\lambda))\,d\nu}{(u_0(\nu)-u_0(\lambda))^2}, \quad \lambda \in (0,\lambda_2),$$

$$(P_{11}^0(\lambda), \psi) = \int_0^1 \frac{\psi(\nu)\,d\nu}{u_0(\nu) - u_0(\lambda)}, \qquad \lambda \in (0, \lambda_2),$$

$$(P_0^0(\lambda), \psi) = \int_0^{\lambda_2} \frac{h_0(\nu)(\psi(\nu) - \psi(\lambda))\,d\nu}{(u_0(\nu) - u_0(\lambda))^2} + \int_{\lambda_2}^1 h_0(\nu)\frac{(\psi(\nu) - Q(\nu, \lambda, \lambda_s, \psi))\,d\nu}{(u_0(\nu) - u_0(\lambda))^2}, \quad \lambda \in (\lambda_2, 1),$$

$$(P_1(\lambda), \psi) = \int_0^{\lambda_2} \frac{\psi(\nu)\,d\nu}{u_0(\nu) - u_0(\lambda)} + \int_{\lambda_2}^1 \frac{(\psi(\nu)(u_0(\nu) - u_0(\lambda)) - h_0(\nu)Q_1^0(\nu, \lambda, \lambda_s, \psi))\,d\nu}{(u_0(\nu) - u_0(\lambda))^2}, \quad \lambda \in (\lambda_2, 1).$$

2. Reduction of the System of Equations (1.3) to a Characteristic Form. Following [2], to reduce Eqs. (1.3) to a characteristic form one should find the eigenfunctionals φ and and eigenvalues $\mathbf{k}$ that obey the equation

$$(\varphi, A\mathbf{f}) = k(\varphi, \mathbf{f}),$$

where the operator A is defined by the equality

$$A(f_1, f_2)^t(\lambda) = (u_0(\lambda)f_1(\lambda) + g\int_0^1 f_2(\nu)\,d\nu\ ,\ h_0(\lambda)f_1(\lambda) + u_0(\lambda)f_2(\lambda))^t.$$

Here $(\varphi, \mathbf{f})$ denotes the action of the functional φ on the trial function $\mathbf{f}$; $\mathbf{f} = (f_1, f_2)^t$ is a sufficiently smooth vector function.

As was shown in [3], the eigenvalues k_i of the discrete spectrum are defined by the equation

$$g\int_0^1 h_0(u_0(\nu) - k_i)^{-2} d\nu = 1, \tag{2.1}$$

which always has two real roots beyond the segment $[\min_\lambda u_0(\lambda), \max_\lambda u_0(\lambda)]$. The operator A has the segment of continuous spectrum $k^\lambda = u_0(\lambda)$ $(0 \leqslant \lambda \leqslant 1)$. Eigenfunctionals φ^i correspond to the eigenvalues k_1 and k_2 satisfying Eq. (2.1). The action of φ^i on the arbitrary smooth function $\mathbf{f}$ $[\mathbf{f} = (f_1, f_2)^t]$ is defined by the formula

$$(\varphi^i, \mathbf{f}) = \int_0^1 f_1(\nu)h_0(\nu)(u_0(\nu) - k_i)^{-2} d\nu - \int_0^1 f_2(\nu)(u_0(\nu) - k_i)^{-1} d\nu \quad (i = 1, 2).$$

For each of the eigenvalues $k^\lambda = u_0(\lambda)$ $[\lambda \in (0, \lambda_2)]$ there are two eigenfunctionals, $\varphi^{11\lambda}$ and $\varphi^{21\lambda}$, which are given by the formulas

$$\varphi^{11\lambda} = (\delta'(\lambda)\,,\, \omega_0(\lambda)\delta(\lambda)), \quad \varphi^{21\lambda} = (gP_{10}^0(\lambda) + \delta(\lambda),\, -gP_{11}^0(\lambda)).$$

For each eigenvalue $k^\lambda = u_0(\lambda)$ $(\lambda \in (\lambda_2, 1))$ there are four eigenfunctionals: $\varphi^{1\lambda}$, $\varphi^{2\lambda}$, $\varphi^{3\lambda}$, and $\varphi^{4\lambda}$ given by the formulas

$$\varphi^{1\lambda} = (\delta'(\lambda) + \delta'(\lambda_s) + 6(\delta(\lambda) - \delta(\lambda_s))(\lambda - \lambda_s)^{-1},\, \omega_0(\lambda)\delta(\lambda) + \omega_0(\lambda_s)\delta(\lambda_s)),$$

$$\varphi^{2\lambda} = ((\delta'(\lambda) - \delta'(\lambda_s))(\lambda - \lambda_s)^{-1},\, (\omega_0(\lambda)\delta(\lambda) - \omega_0(\lambda_s)\delta(\lambda_s))(\lambda - \lambda_s)^{-1}),$$

$$\varphi^{3\lambda} = ((\delta'(\lambda) - \delta'(\lambda_s))(\lambda - \lambda_s)^{-2} + 2(\delta(\lambda) - \delta(\lambda_s))(\lambda - \lambda_s)^{-3},\, (\omega_0(\lambda)\delta(\lambda) + \omega_0(\lambda_s)\delta(\lambda_s))(\lambda - \lambda_s)^{-2}),$$

$$\varphi^{4\lambda} = (gP_0^0(\lambda) + \delta(\lambda)\,,\, -gP_1^0(\lambda)).$$

3. Riemann Invariants. Let us introduce new sought-for functions:

$$R_i = \int_0^1 \frac{h_0(\nu)u(\nu)\,d\nu}{(u_0(\nu) - k_i)^2} - \int_0^1 \frac{h(\nu)\,d\nu}{u_0(\nu) - k_i} \quad (i = 1, 2),$$

for $\lambda \in (0, \lambda_2)$

$$R_{11\lambda} = -u_\nu(\lambda) + \omega_0(\lambda)h(\lambda),$$

$$R_{21\lambda} = g\int_0^1 \frac{h_0(\nu)(u(\nu) - u(\lambda))\,d\nu}{(u_0(\nu) - u_0(\lambda))^2} + u(\lambda) - g\int_0^1 \frac{h(\nu)\,d\nu}{u_0(\nu) - u_0(\lambda)},$$

for $\lambda \in (\lambda_2, 1)$

$$R_{1\lambda} = -(u_\nu(\lambda) + u_\nu(\lambda_s)) + 6(u(\lambda) - u(\lambda_s))(\lambda - \lambda_s)^{-1} + \omega_0(\lambda)h(\lambda) + \omega_0(\lambda_s)h(\lambda_s),$$

$$R_{2\lambda} = (u_\nu(\lambda) - u_\nu(\lambda_s))(\lambda - \lambda_s)^{-1} - (\omega_0(\lambda)h(\lambda) - \omega_0(\lambda_s)h(\lambda_s))(\lambda - \lambda_s)^{-1},$$

$$R_{3\lambda} = (u_\nu(\lambda) + u_\nu(\lambda_s))(\lambda - \lambda_s)^{-2} - 2(u(\lambda) - u(\lambda_s))(\lambda - \lambda_s)^{-3} - (\omega_0(\lambda)h(\lambda) + \omega_0(\lambda_s)h(\lambda_s))(\lambda - \lambda_s)^{-2},$$

$$R_{4\lambda} = \int_0^{\lambda_2} \frac{h_0(\nu)(u(\nu) - u(\lambda))\,d\nu}{(u_0(\nu) - u_0(\lambda))^2} + \int_{\lambda_2}^1 \frac{h_0(\nu)(u(\nu) - Q(\nu, \lambda, \lambda_s, u))\,d\nu}{(u_0(\nu) - u_0(\lambda))^2}$$

$$+\int_0^{\lambda_2} \frac{h(\nu)\,d\nu}{u_0(\nu) - u_0(\lambda)} + \int_{\lambda_2}^1 \frac{(h(\nu)(u_0(\nu) - u_0(\lambda)) - h_0(\nu)Q_1^0(\nu, \lambda, \lambda_s, h))\,d\nu}{(u_0(\nu) - u_0(\lambda))^2}.$$

Lemma. *The quantities R_1 and R_2 are conserved along the characteristics $dx/dt = k_i$ $(i = 1, 2)$. The quantities $R_{i1\lambda}$ $(i = 1, 2)$ are conserved along the characteristics $dx_\lambda/dt = u_0(\lambda)$ $[\lambda \in (0, \lambda_2)]$. The quantities $R_{i\lambda}$ $(i = 1, \ldots, 4)$ are conserved along the characteristics $dx_\lambda/dt = u_0(\lambda)$ $[\lambda \in (\lambda_2, 1)]$.*

4. Solution of the Linear Problem. At the initial instant of time, the Riemann invariants R are the known functions of the initial data. Since these values are conserved along the appropriate characteristics, we obtain an explicit representation of the solution in terms of the Riemann invariants:

$$\begin{aligned} &R_i(x, t) = R_{0i}(x - k_it) \quad (i = 1, 2), \\ &R_{i\lambda}(x, t, \lambda) = R_{0i\lambda}(x - u_0(\lambda)t, \lambda) \quad \text{for} \ \ \lambda \in (\lambda_2, 1) \ \ (i = 1, \ldots, 4), \\ &R_{i1\lambda}(x, t, \lambda) = R_{0i1\lambda}(x - u_0(\lambda)t, \lambda) \quad \text{for} \ \ \lambda \in (0, \lambda_2) \ \ (i = 1, 2). \end{aligned} \tag{4.1}$$

Having solved the system (4.1) with respect to u and h, we find a solution to the initial linear problem.

We reduce the system (4.1) to the equation $f(x, t, \lambda) = u(x, t, \lambda) - u_2(x - u_0(\lambda)t, \lambda)$. The equation for the function f will take the form (for brevity, the arguments x and t are omitted)

$$f(\lambda) - g\int_0^1 \frac{1}{\omega_0}\frac{\partial}{\partial\nu}\left(\frac{f(\nu) - f(\lambda)}{u_0(\nu) - u_0(\lambda)}\right) d\nu = g_1(\lambda), \tag{4.2}$$

where $g_1(\lambda) = g_1(x, t, \lambda)$ is the known function which can be expressed in terms of the initial data:

$$g_1(\lambda) = g\int_0^1 \frac{(h_2(x - u_0(\nu)t, \nu) - h_2(x - u_0(\lambda)t, \nu))\,d\nu}{u_0(\nu) - u_0(\lambda)} - g\int_0^1 \frac{(h_0(\nu)u_{2x}(x - u_0(\nu)t, \nu)t)\,d\nu}{u_0(\nu) - u_0(\lambda)}$$

$$-g\int_0^1 \frac{h_0(\nu)(u_2(x - u_0(\nu)t, \nu) - u_2(x - u_0(\lambda)t, \nu))\,d\nu}{(u_0(\nu) - u_0(\lambda))^2}. \tag{4.3}$$

One can easily check that Eq. (4.2) has a solution of the form $f_{11} = \alpha_1(u_0 - k_1)^{-1} + \alpha_2(u_0 - k_2)^{-1}$ (α_1 and α_2 are arbitrary quantities independent of λ).

Let us seek a general solution of Eq. (4.2) in the form $f = f_1 + f_{11}$, where f_1 satisfies the conditions $f_1(0) = f_1(\lambda_1)$, $f_1(1) = f_1(\lambda_1)$. The fulfillment of these conditions can be ensured by a proper choice of α_1 and α_2. The function f_1 meets the symmetry property $f_1(\lambda) = f_1(\lambda_s)$ $[\lambda \in (\lambda_2, 1)]$. This follows from the equalities (4.1). If the function f_1 is known, the coefficients α_1 and α_2 are determined from the relations $R_i(x,t) = R_{0i}(x - k_i t)$ $(i = 1,2)$:

$$\alpha_i = \left(2\int_0^1 \frac{h_0\,d\nu}{(u_0(\nu) - k_i)^3}\right)^{-1} g s_i \qquad (i = 1,2). \tag{4.4}$$

Here

$$s_i = \int_0^1 \frac{(h_2(x - u_0(\nu)t, \nu) - h_2(x - k_i t, \nu))\,d\nu}{u_0(\nu) - k_i} - \int_0^1 \frac{(h_0(\nu) u_{2x}(x - u_0(\nu)t, \nu)t)\,d\nu}{u_0(\nu) - k_i}$$

$$-\int_0^1 \frac{h_0(\nu)(u_2(x - u_0(\nu)t, \nu) - u_2(x - k_i t, \nu))\,d\nu}{(u_0(\nu) - k_i)^2} - \int_0^1 \frac{(\omega_0^{-1})_\nu f_1 d\nu}{u_0 - k_i} + f_1(\lambda_1)\left(\omega_0^{-1}(\nu)(u_0(\nu) - k_i)^{-1}\Big|_0^1\right).$$

Integrating by parts and changing the variables, we transform Eq. (4.2) into the form

$$\psi(\tau)\left[(\tau - u_{0*}) + g(u_{01} - \tau)^{-1}\omega_{01}^{-1}(u_{01} - u_{0*}) - g(u_{00} - \tau)^{-1}\omega_{00}^{-1}(u_{00} - u_{0*}) - g\int_{u_{00}}^{u_{0*}} \frac{\rho(\tau')\,d\tau'}{\tau' - \tau}\right]$$

$$+g\int_{u_{00}}^{u_{0*}} \frac{\rho(\tau')\psi(\tau')\,d\tau'}{\tau' - \tau} = -f_{1*} + g_1(\tau), \tag{4.5}$$

where $\psi(\tau) = (f_1(\tau) - f_{1*})(\tau - u_{0*})^{-1}$ ($\psi = 0$ with $\lambda = 0$, $\lambda = 1$); τ', τ, and ω_{0s} is the abridged notation for $u_0(\nu)$, $u_0(\lambda)$, and $\omega_0(\lambda_s)$; indices 00, 01, and $0*$ correspond to the values of the functions u_0 and ω_0 for $\lambda = 0$, $\lambda = 1$, $\lambda = \lambda_1$; $f_{1*} = f_1(\lambda_1)$; $\omega_0(\tau) = \tilde{\omega}_0(\tau(\lambda))$; $\omega_{0s}(\tau) = \tilde{\omega}_{0s}(\tau(\lambda))$; $\psi(\tau) = \tilde{\psi}(\tau(\lambda))$; $g_1(\tau) = \tilde{g}_1(\tau(\lambda))$.

The function $\rho(\tau)$, which is discontinuous at the point u_{01}, is specified by the formulas

$$\begin{aligned} \rho(\tau) &= (\tau - u_{0*})\frac{\partial}{\partial\tau}\left(\frac{1}{\omega_0(\tau)}\right) && \text{for } \tau \in (u_{00}, u_{01}), \\ \rho(\tau) &= (\tau - u_{0*})\left(\frac{\partial}{\partial\tau}\left(\frac{1}{\omega_0(\tau)}\right) - \frac{\partial}{\partial\tau}\left(\frac{1}{\omega_{0s}(\tau)}\right)\right) && \text{for } \tau \in (u_{01}, u_{0*}). \end{aligned} \tag{4.6}$$

The function $\rho(\tau)$ has a singularity $\rho = O(|\tau - u_{0*}|)^{-1/2}$ at the point u_{0*}, because, in view of assumptions (1.4), $\lambda = \lambda_1$ $(\tau - u_{0*}) = O((\lambda - \lambda_1)^2)$ in the vicinity of the point and hence $|\omega_0| = |u_{0\lambda} h_0^{-1}| = O(|\lambda - \lambda_1|) = O(|\tau - u_{0*}|^{1/2})$. Integral equation (4.5) is reduced to the Riemann problem

$$\psi_1^+(\tau) = G(\tau)\psi_1^-(\tau) + g_2(\tau), \quad \tau \in (u_{00}, u_{0*}) \tag{4.7}$$

for the function

$$\psi_1(z) = \int_{u_{00}}^{u_{0*}} \frac{\rho(\tau)\psi(\tau)\,d\tau}{\tau - z}$$

in the plane of the complex variable z with a cut along the segment $[u_{00}, u_{0*}]$. Here plus and minus signs correspond to the limiting values of the function for $z \to \tau$ from the upper and lower half-planes; $G(\tau) = (a(\tau) - b(\tau))/(a(\tau) + b(\tau))$; $g_2(\tau) = (\rho(\tau)(-f_{1*} + g_1(\tau)))/(a(\tau) + b(\tau))$; and $g_1(\tau)$ is defined by Eq. (4.3):

$$a(\tau) = (\tau - u_{0*}) + g((u_{01} - u_{0*})(u_{01} - \tau)^{-1}\omega_{01}^{-1} - (u_{00} - u_{0*})(u_{00} - \tau)^{-1}\omega_{00}^{-1}) - g\int_{u_{00}}^{u_{0*}} \frac{\rho(\tau')\,d\tau'}{\tau' - \tau}; \tag{4.8}$$

$$b(\tau) = \pi i g \rho(\tau).$$

Let us extend the boundary condition to the real axis, assuming the function G to be equal to unity at the segments $]-\infty, u_{00}]$, $]u_{0*}, +\infty[$. The Riemann problem under consideration has coefficients that are discontinuous at the point u_{0*} $[G(u_{0*}-0)=-1,\ G(u_{0*}+0)=1]$. Let us seek a solution to the Riemann problem in the class of functions vanishing at infinity and unbounded at the point u_{0*}.

Following the general theory, the problem of the unique solvability of problem (4.7) is solved by calculating its index [4]. The absence of complex roots of the characteristic equation (2.1) and hence the absence the solutions of (1.3) that grow infinitely with time is ensured (following the argument principle) by fulfillment of the condition

$$\frac{1}{\pi}\Delta \arg G = -3, \tag{4.9}$$

where Δ is an increment at the segment $]u_{00}, u_{01}[\cup]u_{01}, u_{0*}[$. If conditions (4.9) and $a^2-b^2\neq 0$ with $z\in(u_{00},u_{0*})$ are fulfilled, the index $æ$ of the Riemann problem (4.7) in the above-mentioned class of solutions equals -1 [3]. When $æ=-1$, problem (4.7) has a unique solution only if

$$\int_{u_{00}}^{u_{0*}}\frac{g_2(\tau)\,d\tau}{X_1(\tau)}=0 \tag{4.10}$$

[X_1 is the canonical solution of problem (4.7)]. The canonical solution of the Riemann problem, which is regarded as a function satisfying the boundary condition and having zeroth order everywhere in the finite part of the plane and the $(-æ)$-order at infinity, has the form

$$X_1(z)=(z-u_{00})(z-u_{01})(z-k_1)^{-1}(z-k_2)^{-1}a(z)=P(z)a(z).$$

Thus, the solution of Eq. (4.5) can be written in the form

$$\psi(\tau)=a(\tau)(-f_{1*}+g_1(\tau))-g(a^2(\tau)-b^2(\tau))P(\tau)\int_{u_{00}}^{u_{0*}}\frac{(-f_{1*}+g_1(\tau'))\rho(\tau')\,d\tau'}{(a^2(\tau')-b^2(\tau'))P(\tau')(\tau'-\tau)},$$

where the value f_{1*} is determined from the solvability condition (4.10):

$$f_{1*}=\int_{u_{00}}^{u_{0*}}\frac{g_1(\tau)\rho(\tau)\,d\tau}{(a^2(\tau)-b^2(\tau))P(\tau)}\Bigg/\int_{u_{00}}^{u_{0*}}\frac{\rho(\tau)\,d\tau}{(a^2(\tau)-b^2(\tau))P(\tau)}. \tag{4.11}$$

Thus, we have proven the following

Theorem. *Let u, u_x, u_t, $u_2(x-u_0(\nu)t,\nu)\in C^{2+\alpha}[0,1]$, h, h_t, h_x, $h_2(x-u_0(\nu)t,\nu)\in C^{1+\alpha}[0,1]$ $(0<\alpha<1)$ and the hyperbolicity conditions $a^2-b^2\neq 0$ and (4.9) hold [a, b from (4.8)]. Then the solution to the Cauchy problem (1.3) has the form*

$$u(x,t,\nu)=f_1(x,t,\nu)+\alpha_1(u_0(\nu)-k_1)^{-1}+\alpha_2(u_0(\nu)-k_2)^{-1}+u_2(x-u_0(\nu)t,\nu),$$
$$h(x,t,\nu)=\omega_0^{-1}(\nu)(u_\nu(x,t,\nu)-u_{2\nu}(x-u_0(\nu)t,\nu))+h_2(x-u_0(\nu)t,\nu)\quad(\nu\neq\lambda_1),$$
$$h(x,t,\lambda_1)=h_0u_{0\nu\nu}^{-1}(\lambda_1)(u_{\nu\nu}(x,t,\lambda_1)-u_{2\nu\nu}(x-u_0(\lambda_1)t,\lambda_1))+h_2(x-u_0(\lambda_1)t,\lambda_1).$$

Here α_1 and α_2 are found from formulas (4.4);

$$f_1(x,t,\nu)=f_1(x,t,\lambda_1)+\psi(x,t,\nu)(u_0(\nu)-u_0(\lambda_1));$$
$$\psi(x,t,\nu)=a(u_0(\nu))(-f_1(x,t,\lambda_1)+g_1(x,t,\nu))-g(a^2(u_0(\nu))$$
$$-b^2(u_0(\nu)))P(u_0(\nu))\int_{u_{00}}^{u_{0*}}\frac{(-f_1(x,t,\lambda_1)+g_1(x,t,\tau))\rho(\tau)\,d\tau}{(a^2(\tau)-b^2(\tau))P(\tau)(\tau-u_0(\nu))},$$

where $P(u_0(\nu))=(u_0(\nu)-u_{00})(u_0(\nu)-u_{01})(u_0(\nu)-k_1)^{-1}(u_0(\nu)-k_2)^{-1}$; a, b are determined from (4.8); $g_1(x,t,\nu)$, from (4.3); $\rho(\tau)$, from (4.6); and $f_1(x,t,\lambda_1)=f_{1*}$, from (4.11).

The formulas give an explicit representation of the solution of the Cauchy problem for the system of equations (1.2) linearized at a shear flow with a nonmonotonic velocity profile.

In conclusion, we gives an example of the velocity profile for which the hyperbolicity conditions are satisfied. Let $u_0 = u_{0*} - ((1/2)y - (u_{0*} - u_{00})^{1/2})^2$, then $\omega_0 = (u_{0*} - u_0)^{1/2}$ for $0 \leqslant \lambda < \lambda_1$, $\omega_0 = -(u_{0*} - u_0)^{1/2}$ for $\lambda_1 < \lambda \leqslant 1$. The function $\chi^+(u_0) = a - b$ has the form

$$\chi^+(u_0) = (u_0 - u_{0*}) - g(u_{0*} - u_{01})^{1/2}(u_0 - u_{01})^{-1} - g(u_{0*} - u_{00})^{1/2}(u_0 - u_{00})^{-1}$$
$$+ \frac{1}{2}(u_{0*} - u_0)^{-1/2} g(\ln((|u_{0*} - u_{00}|^{1/2} + |u_{0*} - u_0|^{1/2})|u_0 - u_{00}|^{-1})$$
$$+ \ln((|u_{0*} - u_{01}|^{1/2} + |u_{0*} - u_0|^{1/2})|u_0 - u_{01}|^{-1})) + \pi i K g(u_{0*} - u_0)^{-1/2},$$

where $K = 1/2$ for $u_0 \in]u_{00}, u_{01}[$; $K = 1$ for $u_0 \in]u_{01}, u_{0*}[$.

Let us describe the behavior of the function $\chi^+(u_0)$ at the segment $]u_{00}, u_{01}[$: $(\chi^+/|\chi^+|)(u_{00} + 0) = -1$, $(\chi^+/|\chi^+|)(u_{01} - 0) = 1$, at the section $]u_{00}, u_{01}[$ $\mathrm{Im}(\chi^+) > 0$, and hence $\Delta \arg \chi^+ = -\pi$ and $\Delta \arg G = -2\pi$ $[G = ((\chi^+)^2/|\chi^+|^2)]$. Furthermore, $(\chi^+/|\chi^+|)(u_{01} + 0) = -1$, $(\chi^+/|\chi^+|)(u_{0*} - 0) = i$. At the segment $]u_{01}, u_{0*}[$ $\mathrm{Im}(\chi^+) > 0$, $\mathrm{Re}(\chi^+) < 0$, and consequently $\Delta \arg \chi^+ = -\pi/2$ and $\Delta \arg G = -\pi$. Thus, the total increment of the argument of the function G at the segment $]u_{00}, u_{01}[\cup]u_{01}, u_{0*}[$ equals (-3π), which means fulfillment of the conditions (4.9).

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REFERENCES

1. V. M. Teshukov, "On hyperbolicity of the long-wave equations," *Dokl. Akad. Nauk SSSR*, **284**, No. 3, 555–559 (1985).
2. V. M. Teshukov, "Long-wave approximation for vortex free boundary flows," in: *Numerical Methods for Free Boundary Problems* (ISNM 99), Birkhäuser Verlag, Basel (1991).
3. V. M. Teshukov and M. M. Sterkhova, "Characteristic properties of the system of equations of a shear flow with a nonmonotonic velocity profile," *Prikl. Mekh. Tekh. Fiz.*, **36**, No. 3, 53–59 (1995).
4. N. I. Muskhelishvili, *Singular Integral Equations* [in Russian], Nauka, Moscow (1968).